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Final Report

November 1965-August 1968

Contract No. AF19(628)-5711

Project No. 5628
Task No. 562803
Work Unit No. 56280301

August 1968

applied mathematics

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Study of the Mathematical Foundations of the Medial Axis Transformation

by

PML Staff

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Abstract

Part One of this report is a brief summary of the research performed under the contract. Part Two presents in 8 sections a study of the skeletal graph G of a silhouette F and some of the relations between the properties of G and the shape of F.

Foreword

The first part of this Final Report summarizes the publications issued under Contract AF19(628)-5711 from inception to July 1968. The second part presents, in Scientific Report form, the results of the last research phase : the study of the skeletal graph.

Table of Contents

Part One

Publications from November 1965 to July 1968

by

PML Staff

1) Technical Memorandum No. 1 (January 1966, 9 Pages)

L. Calabi: The Skeletal Pair Determines the Convex Deficiency.

Essentially absorbed in SR 2 listed below under 5).

2) Technical Memorandum No. 2 (February 1966, 17 Pages)

L. Calabi and W. Hartnett: On "Nice" Sets A.

Background work, ultimately leading to SR 3 listed below under 15).

3) Scientific Report No. 1 (February 1966, 16 Pages)

L. Calabi and W. E. Hartnett: Shape Recognition, Prairie Fires, Convex Deficiencies and Skeletons.

To every closed subset A of the Euclidean plane is associated its convex deficiency D and its skeletal pair (S, q). Extending a known result (A is convex iff $S = \phi$ iff $D = \phi$) one can prove: different sets have the same skeletal pair iff they have the same convex deficiency. Several other results are presented concerning the correspondence $A \rightarrow (S, q)$ and the properties of S and q. The relevance of these notions and theorems for a mathematical model of visual perception is emphasized. The treatment is expository.

Published in the April 1968 issue of Am. Math. Monthly.

4) Technical Memorandum No. 3 (May 1966, 4 Pages)

L. Calabi: Tangents and Half-Tangents of the Skeleton.

Essentially absorbed in SR 3 listed below under 15) and in Part Two of this report.

5) Scientific Report No. 2 (November 1966, 24 Pages)

L. Calabi and W. E. Hartnett: A Generalization of the Motzkin Theorem.

A figure A in the Euclidean plane is a compact set whose closed convex hull $C(A)$ has a non empty interior; a ball of support for A is a closed ball which has points of A on the boundary but not in the interior. For each figure A, let $C(A) - A$ denote the convex deficiency of A and let (S, q) denote the skeletal pair of A where S is the set of centers of maximal balls of support for A and $q(x)$ is the distance from x to A for $x \in S$. The following statements are proved: (1) Two figures have equal convex deficiencies iff they have equal skeletal pairs. (2) (Motzkin's Theorem) A figure is convex iff its skeleton is empty. (3) A figure is uniquely determined by its closed convex hull and its skeletal pair. (4) A figure with empty interior is uniquely determined by its skeletal pair.

6) Technical Memorandum No. 4 (December 1966, 6 Pages)

L. Calabi and W. E. Hartnett: Skeletons of Gray-Scale Pictures.

How to modify the notion of skeletal pair to represent gray-scale, and not only black-and-white, pictures.

7) Technical Memorandum No. 5 (January 1967, 27 Pages)

W. E. Hartnett: A Preliminary Discussion of Mathematical Models and their Suitability.

Discussion of some basic features of mathematical models, as exemplified by the skeletal pair.

8) Technical Memorandum No. 6 (February 1967, 9 Pages)

L. Calabi: On the Curvature of a Skeleton.

Essentially absorbed in Part Two of this report.

9) Technical Memorandum No. 7 (April 1967, 34 Pages)

W. E. Hartnett: A Study of Approximation for Skeletal Pairs: Selection of Adequate Topologies.

Together with TM 12 listed below under 14), this Technical Memorandum presents the initial results in a study of "natural" Topologies on the set of figures and on the set of skeletal pairs for which the various "natural" correspondences would be continuous.

10) Technical Memorandum No. 8 (June 1967, 6 Pages)

L. Calabi: On Nice Sets, II

Background work, ultimately leading to SR 3 listed below under 15).

11) Technical Memorandum No. 9 (June 1967, 8 Pages)

L. Calabi: On "Smooth" Boundaries

Background work, ultimately leading to SR 3 listed below under 15).

12) Technical Memorandum No. 10 (July 1967, 13 Pages)

L. Calabi and W. E. Hartnett: Mathematical Analysis of a Process of Shape Recognition.

Summary of work done at PML on the subject, since 1964 (32 references).

' Technical Memorandum No. 11 (August 1967, 7 Pages)

L. Calabi and W. E. Hartnett: A Motzkin-Type Theorem for Closed Nonconvex Sets.

A concise proof of an extension of the basic theorem presented in SR 2, listed above under 5).

To appear in Proc. A.M.S.

14) Technical Memorandum No. 12 (August 1967, 21 Pages)

W. E. Hartnett: Implications and Non-implications About Shape.

See TM 7 listed above under 9).

15) Scientific Report No. 3 (December 1967, 28 Pages)

L. Calabi and J. A. Riley: The Skeletons of Stable Plane Sets.

Necessary and sufficient conditions are formulated for the skeleton of a set A to be, essentially, a "well-behaved" graph. Differentiability properties of that graph and of the quench function are established.

16) Technical Memorandum No. 13 (January 1968, 29 Pages)

W. E. Hartnett: A Preliminary Study of Oriented Skeletal Graphs.

Background work, ultimately leading to Part Two of this report.

17) Technical Memorandum No. 14 (February 1968, 17 Pages)

W. E. Hartnett: A Definition of Skeletal Graph of a Compact Set.

Essentially absorbed in Part Two of this report.

18) Technical Memorandum No. 15 (April 1968, 13 Pages)

L. Calabi: Shape Interpretations of the Natural Vertices of the Skeletal Graph.

Essentially absorbed in Part Two of this report.

19) Technical Memorandum No. 16 (May 1968, 7 Pages)

<u>L. Calabi</u>: Fiala's Results on the Skeleton of a Curve in a Riemannian Plane.

Quick presentation of the results of the title, and comparison with our theory.

20) Technical Memorandum No. 17 (May 1968, 15 Pages)

<u>William E. Hartnett</u>: Partial Bibliography for Shape Recognition.

A list of 49 papers and books, none by the PML group, on topics relevant to the study of the skeletal pair.

21) Technical Memorandum No. 18 (June 1968, 16 Pages)

<u>W. E. Hartnett</u>: Curved Skeletal Graphs.

Background work, ultimately leading to Part Two of this report.

Part Two

A Study of the Skeletal Graph

by

L. Calabi

1. Introduction

Assuming the reader generally conversant with previous PML work on the general subject, this report studies the notion of skeletal graph G of a silhouette F and shows that certain shape attributes of F are easily describable in terms of G.

The most interesting outcome for application purposes is the possibility of studying F piecemeal by considering each vertex and each edge of G separately. Theorems 4.3, 6.4 and 7.6, for instance, show that such "local" investigations may furnish non trivial information on the boundary of F, and hence on F itself.

From a theoretical point of view the smoothness properties of each edge of G is an interesting consequence of our general assumptions. Even more interesting are Theorems 4.3 and 4.4, which extend to the non-differentiable case a result so far known only for twice continuously differentiable boundaries. That is a further indication that the "nearest points map" and its related notions are indeed a powerful tool to study geometric properties classically considered the domain of differential geometry (cfr. [5]).

The definition of skeletal graph G is given in Section 2, together with the general properties assumed to hold throughout the report. Sections 3 and 4 present a study of the vertices of G, though they contain also results on points of order 2, not vertices, to be used later. The decomposition of F introduced in Section 5 is logically helpful and geometrically natural, even if visually too complex

to be appealing. Sections 6 and 7 study the edges of G and some of their influence on the boundary of F. Finally, in Section 8, the convexity of F is shown equivalent to several properties of π or of G.

Some of the spadework leading to the definition of skeletal graph was done by Dr. W.E. Hartnett; for it, and for many helpful discussions, I am gratefully indebted to him.

2. The Skeletal Graph

Let A be a closed set in the plane and let (S, g) be its skeletal pair. We assume that:

(1) S is closed in the complement of A, bounded and connected;

(2) All the points of S have finite order, and only finitely many have an order different from 2.

Then we know that the closure $\bar{S}$ is a disjoint union of finitely many open arcs and their endpoints, that we shall call natural edges and natural vertices respectively. A point x of a natural edge E will be called an extremal vertex iff it has a neighborhood U in E such that $g(x') \geq g(x)$ (or $g(x') \leq g(x)$) for all $x' \in U$ and there is a sequence of points $x_n \in E$, $x_n \to x$ with $g(x_n) > g(x)$ (or $g(x_n) < g(x)$). Each natural edge is then the disjoint (possibly trivial) union of its extremal vertices and of open arcs, called monotone edges since g is either constant or strictly monotone on them. We shall require:

(3) The number of extremal vertices in S is finite.

If x belongs to a natural edge, it has order 2 in S and hence πx has

exactly two (connected) components; we shall say that such a point x is a jump vertex iff πx has more than two points (i.e., at least one of the components of πx is a non-degenerate arc of circle). Our next assumption is then:

(4) The number of jump vertices in S is finite.

Denoting by V the set of those points of $\bar{S}$ which are either natural, or extremal, or jump vertices, and assuming $V \neq \phi$ we introduce on the set $\bar{S}$ a structure of graph by considering V as the set of vertices and by introducing as edges the (arcs which are) components of $\bar{S} \setminus V$. The set $\bar{S}$ with that structure we shall denote by G and call the skeletal graph of A. We will extend q to G by setting $q(x) = 0$ if $x \in G \setminus S$. (For the case $V = \phi$, see Theor. 5.5 below).

We will denote by F° that component of the complement of A which contains S and set $F = \bar{F}^{\circ}$. We assume now:

(5) The boundary C of A and that of F are equal. Moreover every point y of C is a local separating point of C.

(A set Y, or a point y, is said to be locally separating if, for some open neighborhood $\mathcal{U}$, $\mathcal{U} \setminus Y$ or $\mathcal{U} \setminus \{y\}$ is not connected.) We could then say that F is the silhouette of G and that G is the graph of F. Observe that for every closed disc D containing F in its interior, $D \setminus F^{\circ}$ is a stable set (in the sense of [1]), having (S, q) as skeletal pair and hence G as skeletal graph.

Our five assumptions are probably not independent. We will use them all in the sequel. The formulation of an irredundant set of

axioms would require an effort out of proportion with the scope of this study.

In any graph, if n_i denotes the number of vertices of order i, then $\frac{1}{2}\sum i n_i$ is the number of edges. Further, if the graph is connected, $n_1 - 2 \le \sum_{i>2} i n_i$, showing that if there are at least 3 vertices of order 1, then there certainly are vertices of order 3 or more. Finally again if the graph is connected the number $\sum n_i - \frac{1}{2}\sum i n_i + 1$, called the connectivity of the graph, is the number of its "windows" as well as the largest number of edges which may be removed without disconnecting the graph. In our case the connectivity of G is also the number of bounded components of the complement of F, that is the number of "holes" in F.

If $x \in G$, we shall call <u>branches (of G) at x</u> the edges incident with x, if x is a vertex; and otherwise, if x belongs to the edge E, the two components of $E \setminus \{x\}$. To each branch B_i at x, we know from [1] that there correspond

- a half-tangent $\ell_i(x)$ to G at x (more precisely to $\bar{B}_i$ at x);
- two points $y_i, y_i' \in \pi x$ such that $[y_i, x], [y_i', x]$ are the side of the π-sector containing B_i;
- a number $\alpha_i(x)$ measuring the angle between $\ell_i(x)$ and $[x, y_i]$, as well as the angle between $\ell_i(x)$ and $[x, y_i']$.

We further remember that the number of branches at x, denoted $o(x)$, is not only the number of π-sectors of vertex x, but also the order of x in G and the number of components of $\pi' x$.

If $x' \in B_i$, sufficiently close to x, then there exists a branch B_i' at x' contained in B_i. Let $\ell_i(x')$, $\alpha_i(x')$ and y_i'', y_i''' be the corresponding elements. When x' tends to x along B_i, y_i'' and y_i''' tend respectively to y_i and y_i', showing that $\ell_i(x') \to \ell_i(x)$ and $\alpha_i(x') \to \alpha_i(x)$. We express this as follows:

Lemma 2.1 Let E be an edge and give to $\bar{E}$ an orientation. For each point $x \in \bar{E}$ but the last, let $\ell(x)$ be the right half-tangent to $\bar{E}$ at x; and let $\alpha(x)$ be half the measure of the angle of the corresponding π-sector. Then the mappings ℓ and α are continuous in E and continuous from the right at the first point of $\bar{E}$.

We topologize the set of rays and the set of lines in the usual fashion.

3. Points of Order 2

Since, under our general assumptions, πx cannot have infinitely many components, we may state without proof:

Lemma 3.1 For any point $x \in S$, $\sum \alpha_i(x) \leq \pi$, with equality holding iff πx is a finite set. Moreover the arc length of πx is given by $2\varphi(x)(\pi - \sum \alpha_i(x))$.

(The symbol π is here used to denote the nearest points map, as in πx, as well as the usual constant 3.14... ; no confusion should arise in the context.)

Theorem 3.2 If $x \in F$, of the following three statements the first implies the second, and the second and third are equivalent:

(a) $x \in G$ is a jump vertex;

(b) x is the center of a circle having a (non-degenerate) arc in common with C ;

(c) $x \in G$ and $\sum \alpha_i(x) < \pi$.

Moreover, if x has order 2 in G, then the three statements are equivalent.

Proof Immediate from the definitions and Lemma 3.1.

Thus, if we know that x is a jump vertex, we also know that C has one or more arcs of a circle centered at x ; and if we know the $\alpha_i(x)$, we know the total central angle of those arcs; and if we know $q(x)$ we know the circle; and if we also know the half-tangents $\ell_i(x)$ we know the arcs themselves and hence a part of C .

We turn now to the study of the extremal vertices. Remember that on each edge, and hence also on each branch, q is monotone. We agree to orient the branches at x away from it.

Lemma 3.3 If $\alpha_i(x) > \pi/2$, then q is strictly increasing on B_i. Similarly, if $\alpha_i(x) < \pi/2$, then q is strictly decreasing on B_i.

Proof With the notation of Fig. 3.1, $q(x') \geq d(x', z)$ and $d(x', z)^2 = q(x)^2 + d(x, x')^2 - 2q(x)\,d(x, x')\cos\beta$. For x' sufficiently close to x, γ' is close to γ and so $\cos\beta < 0$, yielding $q(x') > q(x)$. Since q is strictly monotone if not constant, we obtain the first statement. The second statement is established in a similar fashion (see also [2]).

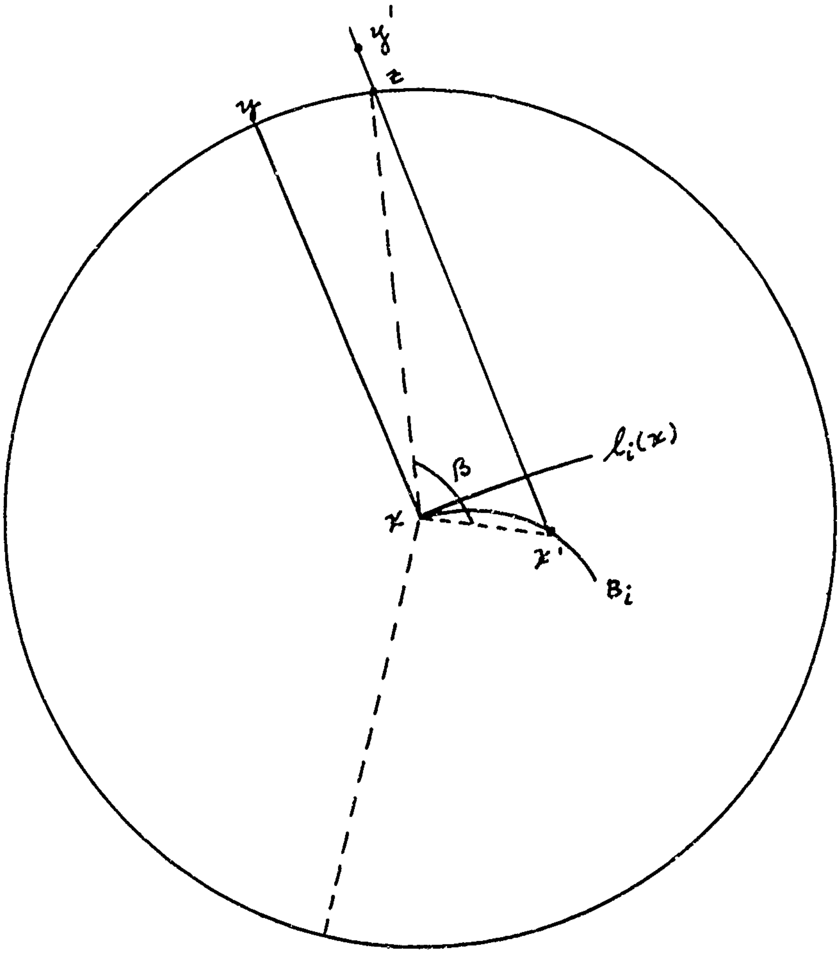


Figure 3.1

Lemma 3.4 If g is strictly increasing (or strictly decreasing, or constant) on B_i, then $\alpha_i(x) \geq \pi/2$ (or $\alpha_i(x) \leq \pi/2$, or $\alpha_i(x) = \pi/2$ respectively).

Proof From Lemma 3.3. Notice that $\alpha_i(x) = \pi/2$ is possible in all cases.

Theorem 3.5 Let $x \in G$ be a point of order $o(x) \geq 2$ at which g has a minimum. Then $o(x) = 2$, $\alpha_1(x) = \alpha_2(x) = \pi/2$ and x is an extremal but not a jump vertex.

Proof From Lemmas 3.1 and 3.4 and from Theorem 3.2.

Theorem 3.6 Let $x \in G$ be an extremal vertex at which g has a maximum. Then, if x is not a jump vertex, $\alpha_1(x) = \alpha_2(x) = \pi/2$.

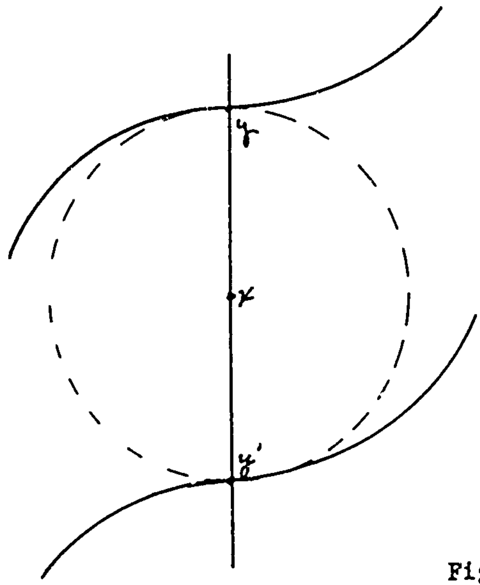
Proof From Lemmas 3.1 and 3.4.

Figures 3.2 to 3.5 illustrate the fact that extremal vertices do not, by themselves, express shape properties of G. Notice that in all four examples $\alpha_1(x) = \alpha_2(x) = \pi/2$.

If x is a point of order 2, not a jump vertex, then $\alpha_1(x) + \alpha_2(x) = \pi$ and hence $\ell_1(x)$, $\ell_2(x)$ are collinear. The line to which they belong is the tangent $t(x)$ to G at x. (Notice that such a tangent may also exist at jump vertices.)

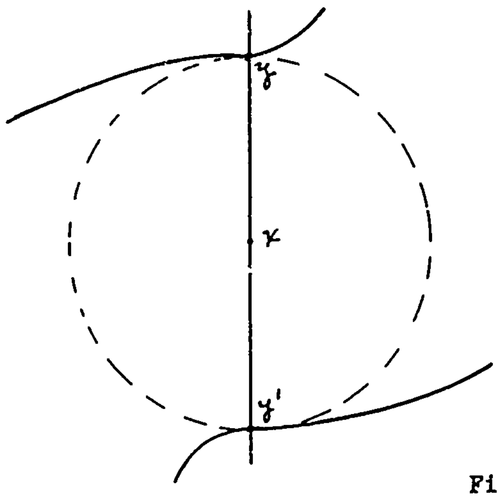
Theorem 3.7 Let $x \in G$ be a point of order two, not a jump vertex. Then G has a tangent $t(x)$ at x and the mapping t is continuous.

Proof From above and Lemma 2.1.



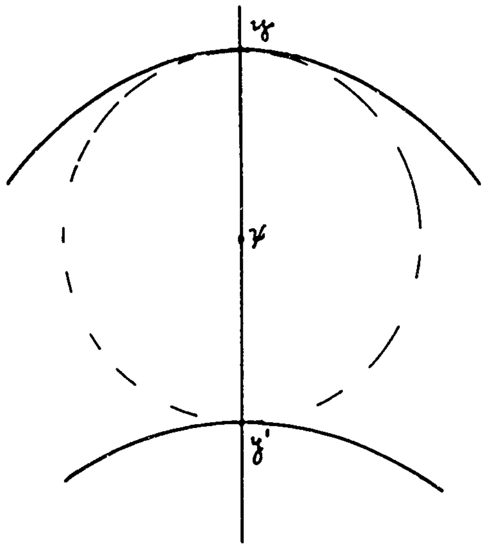
g has a maximum at x

Figure 3.2



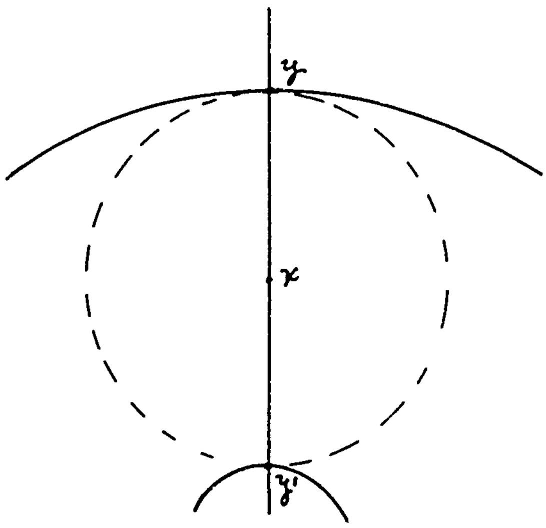
g has a minimum at x

Figure 3.3



q has a maximum at x

Figure 3.4



q has a minimum at x

Figure 3.5

For jump vertices obvious geometric considerations yield:

Theorem 3.8 Let $x \in G$ be a jump vertex and let α_0 be the measure of the angle between $\ell_1(x)$ and $\ell_2(x)$. Then πx consists of two arcs whose lengths are $g(x)(\alpha_0 - \alpha_1(x) - \alpha_2(x))$ and $g(x)(2\pi - \alpha_0 - \alpha_1(x) - \alpha_2(x))$.

At any point $x \in G$ we now define $\alpha(x) = \max \alpha_i(x)$, to obtain:

Theorem 3.9 Let $x \in G$ be a point of order 2, not a jump vertex. Then α is continuous at x.

Proof If x is not a vertex, this follows at once from Lemma 2.1. If x is a vertex, then again by Lemma 2.1, α is continuous from the right on each edge to which it is incident, when oriented away from it. Hence α is continuous.

4. Points of Order Different from 2.

We begin with two results which do not need the general assumptions of Section 2:

Lemma 4.1 Let P be a component of πx, $x \in F$ and assume that P locally separates C. Then, given any neighborhood U of P in C, there is a circle C_0 of center x such that $C_0 \cap U$ has at least two components.

Proof Given U, there is a neighborhood V of P with $U \supset V$ and $V \setminus P$ not connected; and there is a real number $d > g(x)$ such that the circle C_0 of center x and radius d intersects at least two components V_1, V_2 of $V \setminus P$. Indeed, given V_1, V_2 we may set

$$2d = g(x) + \min\{\sup\{d(x,z),\ z \in V_1\},\ \sup\{d(x,z),\ z \in V_2\}\}.$$

Lemma 4.2 Let πx, $x \in G$ have $o(x) < \infty$ components, each locally separating C. Then, given any neighborhood $\mathcal{U}$ of πx in C, there is a circle C_0 of center x such that $C_0 \cap \mathcal{U}$ has at least $2 \cdot o(x)$ components.

Proof From above.

Theorem 4.3 Let $x \in G$ be a vertex of order 1 with πx reduced to a single point y (possibly $x = y$). Then g reaches a minimum at x; and in any neighborhood $\mathcal{U}$ of y in C there are four points belonging to a circle.

Proof That $g(x)$ is a minimum follows from Lemma 3.3. To prove the second statement, let x_n be points on the branch B at x, converging to x. Then πx_n has at least two points y_n, y'_n which may be chosen as follows (see [1]); $y_n \to y$, $y'_n \to y$ and, $[x_n, y_n], [x_n, y'_n]$ are the sides of a π-sector whose boundary contains y. Further, given the neighborhood $\mathcal{U}$, for n sufficiently large y_n and y'_n belong to different components of $\mathcal{U} \setminus \{y\}$, because every point of C is locally separating. Applying then Lemma 4.2 to the corresponding x_n we obtain the desired result.

We have the following partial converse:

Theorem 4.4 Let C_n, $C_n \neq C_m$ if $n \neq m$, be circles converging to a circle C_0 and assume that $C_n \cap C$, $n > 0$ has at least four points

y_n, y'_n, y''_n, y'''_n such that the four sequences (y_n), (y'_n), (y''_n) and (y'''_n) all converge to the same point y. If y is the only point of $C_0 \cap C$, then the center x of C_0 is a vertex of G of order 1 at which g reaches a minimum.

Proof Since $C_0 \cap C$ has only one point, C_0 is a circle of support to A at y. Assume that there is a larger circle C'_0 of support to A at y; then, for each n, the points y_n, y'_n, y''_n and y'''_n lie on or outside C'_0 and hence $C_n \to C'_0$. That is $C_0 = C'_0$ and C_0 is maximal. If its radius is zero, $x = y \in G \setminus \bar{S}$, otherwise $x \in S$. Hence the theorem.

Observe that points y with the properties of Theorem 4.3 or Theorem 4.4 have been called vertices of C with respect to the family of circles; and that, if C is a sufficiently smooth curve, they are but the points of C at which the curvature has an extremum ([3], Section 4.1.1). Our two last theorems then extend to the non-differentiable case a well known result (cfr. [4]); they are thus two more steps in the direction started by Bouligand almost forty years ago (cfr. [5]). See also Theorem 10 of [9].

For completeness we give the next result characterizing those vertices of G which are not in S:

Theorem 4.5 For $x \in F$ the following statements are equivalent:

(a) $x \in G \setminus S$;

(b) x is a natural vertex of G and $g(x) = 0$;

(c) $x \in C$ and the reach of A at x is zero.

Proof The equivalence of (a) and (b) follows from the definition of G . The equivalence of (a) and (c) is established already in [6]. As Fig. 4.1 illustrates, the order of x need not be 1.

We turn now to the points of G of order at least 3 and we start with:

Theorem 4.6 Let $x \in G$ have order at least 3. Then g is strictly increasing on at most one branch at x .

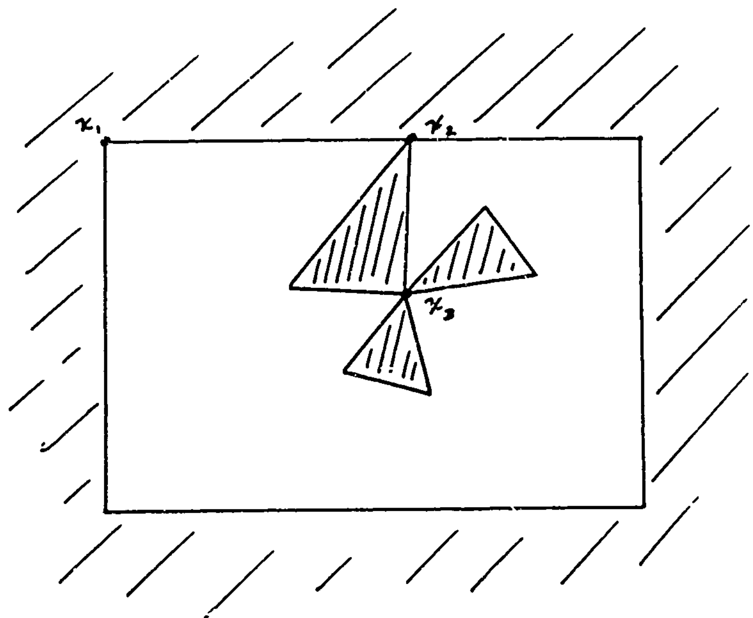
Proof From Lemmas 3.1 and 3.3.

For a more geometric study we need some new notions. Given $x \in F^\circ$ and $y_1, y_2, \ldots, y_r \in C$, $r \geq 2$, assume $[x, y_i] \subset F$; then each component W of $F \setminus \cup [x, y_i]$ is called a wedge of vertex x ; if $[x, y_i] \subset \bar{W}$, then $[x, y_i]$ is called a side of W . We have:

Lemma 4.7 If c is the number of components of C containing points y_i, there are $r-c+1$ wedges of vertex x and sides $[x, y_i]$, $i = 1, 2, \ldots, r$.

Proof If $c > 1$ and $r_j > 0$ is the number of points y_i in the j[th] component C_j of C , then there are $r_j - 1$ wedges whose boundary intersects C only in C_j . There is also one wedge whose boundary intersects all c components, yielding a total of $r-c+1$. If $c = 1$, the lemma is obvious.

Observe that, under the assumption of the previous lemma, there is a wedge having at least c , but not more than $2c$ sides. If all the sides of a wedge W have equal length λ , consider the disc D of center x and radius λ ; W is said to be a sector iff it contains each



A : the shaded set

x_i : natural vertex of G , with $o(x_i) = i$

$x_i \in G \setminus S$, $g(x_i) = 0$

Vertices of G , not in S , of various order

Figure 4.1

component of $D \setminus \bigcup [x, y_i]$ with which it has common points. Further a sector is termed <u>normal</u> iff in each open side (x, y_i) there is a point z_i with $y_i \in \pi z_i$. (Notice that if $[x, y_i]$ is orthogonal to C at y_i, then such a point z_i will exist.)

<u>Lemma 4.8</u> For $x \in F$ the following two statements are equivalent:

(a) πx has at least two points in one component of C ;

(b) there are normal sectors of vertex x .

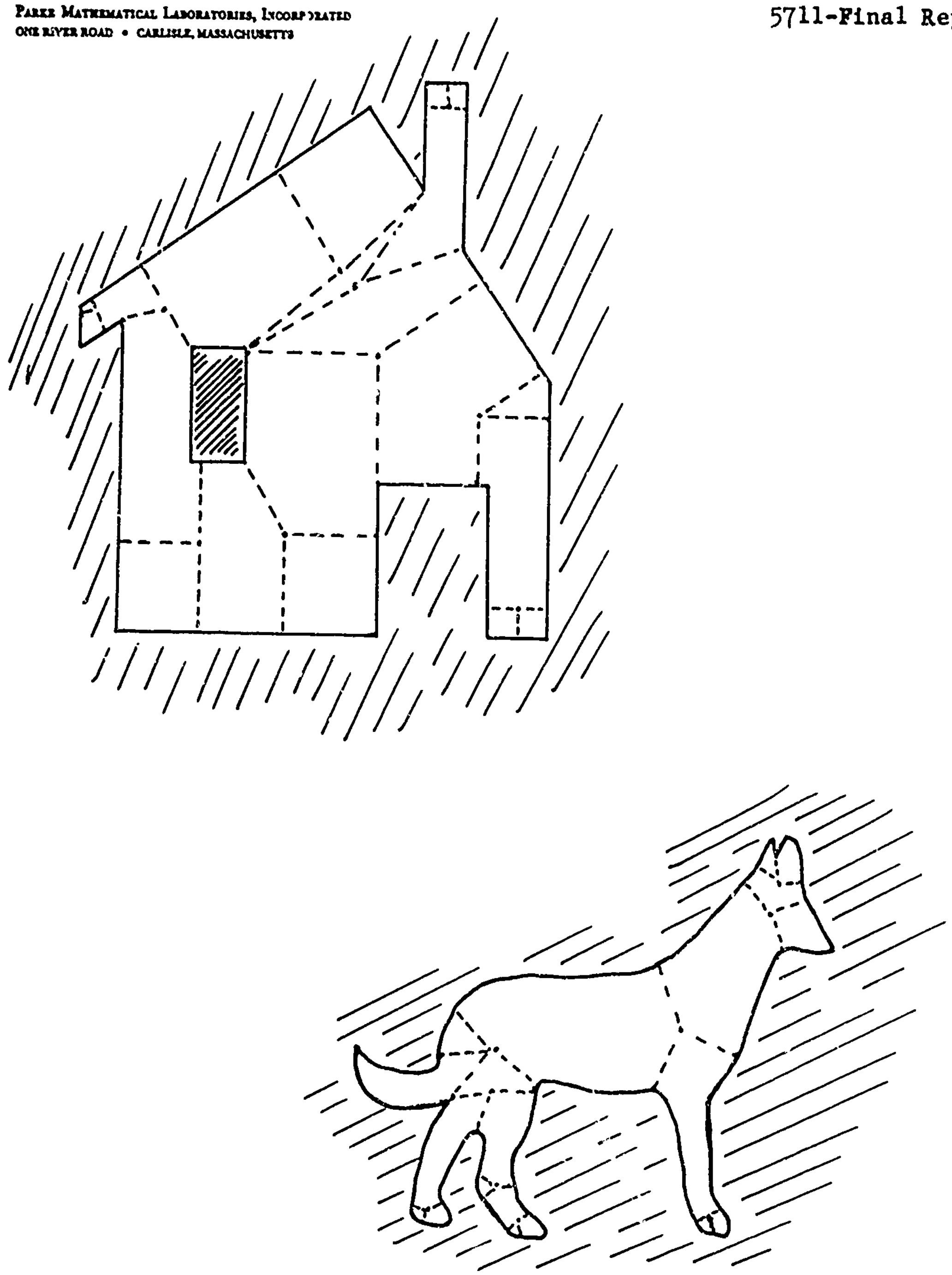
<u>Proof</u> If $y_1, y_2 \in \pi x$ belong to a component of C , then clearly $F \setminus \bigcup [x, y_i]$ contains a normal sector. The converse follows at once from Lemma 4.7.

We say that a normal sector is <u>minimal</u> if no other normal sectors are contained in it. We then have the following characterization, where we denote by $c(x)$ the number of components of C having points in common with πx :

<u>Theorem 4.9</u> The point $x \in G$ is a point of order $o(x) \geq 2$ iff there are $o(x) - c(x) + 1$ minimal normal sectors of vertex x .

<u>Proof</u> From Lemma 4.7 and 4.8. The minimal normal sectors are but the π-sectors at x.

Notice that if πx contains an arc, then there are infinitely many normal sectors of vertex x , none minimal. By restricting the last theorem to the case $o(x) > 2$ we obtain a characterization of the natural vertices of order $\neq 1$.



Two natural decompositions

Figure 5.1

Let us denote by H_1, H_2 the two components of πW and set, for $x \in E \subset W$, $f_i(x) = H_i \cap \pi x$.

Lemma 5.3 Each mapping f_i, $i = 1, 2$ is a continuous mapping of E onto H_i. If f_i is $1-1$, then it is an homeomorphism.

Proof Clearly f_i is onto; its continuity follows from the upper semi-continuity of π. To establish the last statement fix i, set $g = f_i^{-1}$ and let $y_n \to y$ in H_i, $x_n = g(y_n)$. We may assume that (x_n) converges to some point x: we have to prove $x = g(y)$. By the upper semi-continuity of π, we know that $y \in \pi x$ since $y_n \in \pi x_n$. If $x \in A$, then $y = x$ is a vertex of G and hence $y \notin H_i$. Thus $x \notin A$; but S is closed in the complement of A, $x_n \in S$ and consequently $x \in S$. That is, $f_i(x) = y$ or $g(y) = x$.

Lemma 5.4 The boundary C is a finite union of arcs, intersecting only at their endpoints.

Proof From Lemmas 5.1 and 5.3.

If there are no vertices, F has no natural decomposition; or, better, F° is the only section of F. The results given above still apply and moreover we have:

Theorem 5.5 The following two statements are equivalent:

(a) the set V of vertices is empty;

(b) S is a closed simple curve on which $g(x) = g_0$ is constant.

Moreover either statement implies: C is the union of two parallel curves at distance $2g_0$.

Proof If (a) holds, there are no extremal values for g, hence g is constant; and all points of S have order 2, hence S is a closed simple curve. Conversely, if (b) holds there are no natural or extremal vertices; and $\alpha_1(x) = \alpha_2(x) = \pi/2$ for each $x \in S$. Thus there are also no jump vertices. The last statement of the theorem then follows.

6. A Study of the Edges

Given an edge E of G, let us denote by $a = a(E)$ and $b = b(E)$ its vertices (= endpoints of $\bar{E}$). Remember that $a = b$ is possible. We orient E from a to b and assume that g is then non decreasing. For $x \in E$, we let $\alpha(x) = \max \alpha_i(x)$, as in Theorem 3.9; further $\alpha(a) = \lim \alpha(x)$ and similarly for $\alpha(b)$. Then α is continuous on $\bar{E}$. Similarly for the tangent $t(x)$ (see Theor. 3.7); we orient $t(x)$ so that, if x_1 follows x on E, the oriented straight line from x to x_1 tends to $t(x)$ when $x_1 \to x$.

Theorem 6.1 Every closed arc E_0 contained in $E \cup \{b\}$ is rectifiable. Moreover, if $g(a) > 0$, then $\bar{E}$ is also rectifiable.

Proof From the proof of Theorem D of [1] we know that at every point x of $\bar{E} \cap S$ the paratingent of $\bar{E}$ is reduced to one line, namely $t(x)$. Our result follows then, e.g., from [5], Section 80. The assumption $g(a) > 0$ is essential for the validity of the second statement: indeed $\bar{E}$ with $g(a) = 0$ could be, for instance, a spiral (F^* could then be the region bounded by two non crossing spirals with the same vertex).

It may be interesting to remember here a result of [8]. Let

$C_\rho = \{x \in F,\ d(x, A) = \rho\}$: then, for almost all ρ for which $C_\rho \neq \phi$, C_ρ is a union of rectifiable curves (see also [10]). Notice that C_ρ is a level curve of the function $d(\cdot, A)$, while S is its "crest" or "divide". Under our general assumption we will prove below that $C_o = C$ itself is often rectifiable.

We first give two lemmas, valid without our general assumptions, to obtain quantitative information on the distance between points on the skeleton and corresponding points on C.

<u>Lemma 6.2</u> Given $x, x_1 \in F^\circ$, $y \in \pi x$, $y_1 \in \pi x_1$ and with the notation of Fig. 6.1, we have

(a) $$\bar{d}^2 = (d - r(x)\cos\gamma + r(x_1)\cos\gamma_1)^2 + (r(x)\sin\gamma - r(x_1)\sin\gamma_1)^2$$
$$= d^2 + r(x)^2 + r(x_1)^2 - 2d(r(x)\cos\gamma - r(x_1)\cos\gamma_1) - 2r(x)r(x_1)\cos(\gamma_1 - \gamma)$$

(b) $d \geq r(x)\cos\gamma - r(x_1)\cos\gamma_1$ with equality iff $y = y_1$.

<u>Proof</u> Statement (a) is elementary and holds in any quadrilateral. Statement (b) uses the fact that $r(x) = d(x, A)$. Thus

$$r(x)^2 \leq d(x, y_1)^2 = r(x_1)^2 + d^2 + 2r(x_1)d\cos\gamma_1$$
$$r(x_1)^2 \leq d(x_1, y)^2 = r(x)^2 + d^2 - 2r(x)d\cos\gamma .$$

Adding and rearranging we obtain (b).

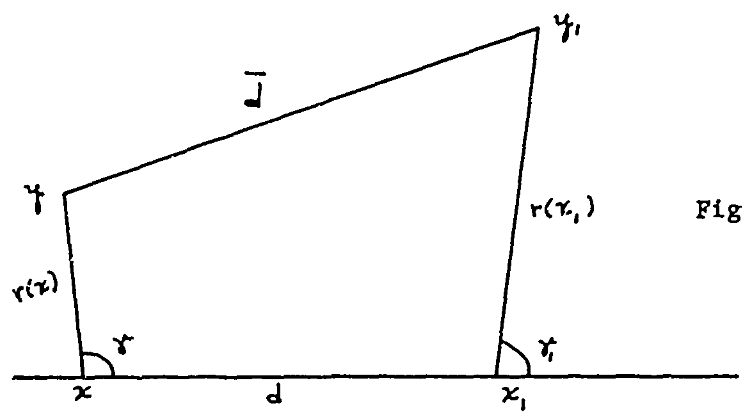


Figure 6.1

<u>Lemma 6.3</u> Under the assumptions and with the notation of Lemma 6.2, let also $r(x_1) \geq r(x)$. Then

$$\bar{d} \leq d + r(x_1) - r(x) \leq 2d \quad \text{if} \quad \gamma_1 \geq \gamma$$

$$\bar{d} \leq d + r(x_1) - r(x) + r(x)(\gamma - \gamma_1) \leq 2d + r(x)(\gamma - \gamma_1) \quad \text{if} \quad \gamma \geq \gamma_1$$

<u>Proof</u> That $r(x_1) - r(x) \leq d$ is a general property of the function r. The other inequalities may be established in every quadrilateral. To prove the first we have (see Fig. 6.2)

$\bar{d} \leq d(y_1, z)$ by construction of z and the assumption $\gamma_1 - \gamma > 0$

$d(y_1, z) \leq d + r(x_1) - r(x)$ by the triangular inequality.

The second statement of the lemma may be established in a similar fashion.

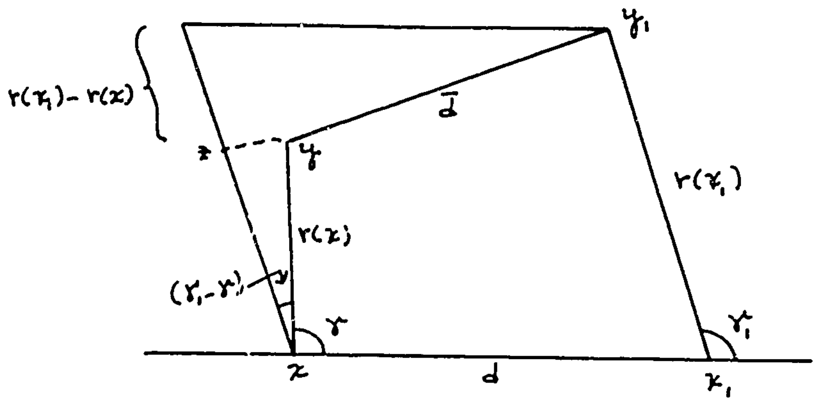


Figure 6.2

Given the edge E and $x \in \bar{E}$ denote by y the point of πx on the left of $t(x)$ and (for the endpoint of $\bar{E}$) such that the angle from $t(x)$ to $[x, y]$ is $\alpha(x)$. Call <u>projecting ray</u> $\rho(x)$ the ray $[x, y \rightarrow)$ and

denote by f the mapping $x \to y$. As customary, we say that f <u>shrinks on</u> (or <u>expands on</u>) a subarc E_0 of E iff $d(f(x), f(x_1)) \leq d(x, x_1)$ (or $(d(f(x), f(x_1)) \geq d(x, x_1))$ for any two points of x, x_1 of E_0.

If x_2 follows x_0 on $\bar{E}$, let $E(x_0, x_2)$ denote the closed subarc of which x_0, x_2 are endpoints and set $H(x_0, x_2) = f(E(x_0, x_2))$. We shall say that $H = H(x_0, x_2)$ is <u>E-concave</u> (or <u>E-convex</u>) (see Fig. 6.3) iff $H \cup [y_0, y_2]$ with $y_0 = f(x_0)$, $y_2 = f(x_2)$ is the boundary of a convex set B and, for any $x \in E(x_0, x_2)$, $d(x, H) \leq d(x, [y_0, y_2])$ (or $d(x, H) \geq d(x, [y_0, y_2])$). The terminology wants to suggest the apparent shape of H, when seen from some point of $E(x_0, x_2)$.

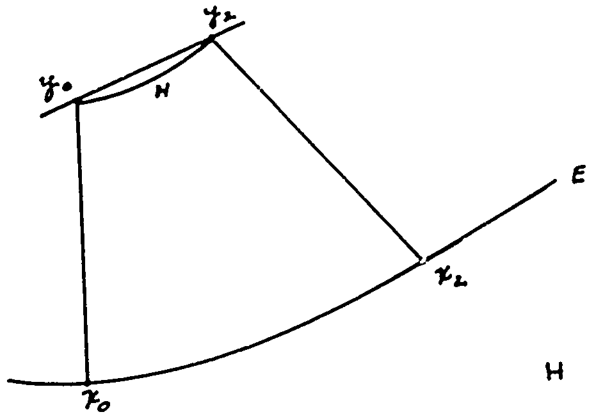
Finally we shall denote by $\beta(x)$ the angle from a fixed, arbitrary directed line ℓ to $p(x)$. Notice that $\beta(x) - \alpha(x)$ is then the angle from ℓ to $t(x)$. Thus β is a continuous function of x. With that terminology we have the following results which reproduce "locally" a known result [11] and which enable us to know something about the shape of F. Similar results may be easily formulated for that component of $\pi E(x_0, x_2)$ which lies to the right of E. Care must be taken however to consider the orientations involved and hence the signs of the angles and the corresponding inequalities.

<u>Theorem 6.4</u> With the notation just introduced assume that $g(x_0) > 0$. Then the following three statements are equivalent:

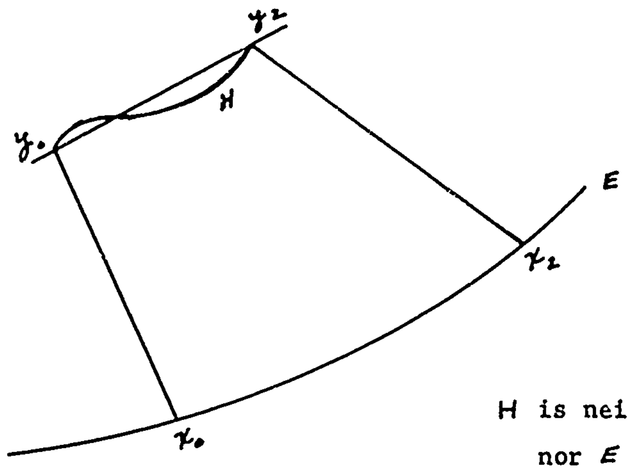
(a) f shrinks on $E(x_0, x_2)$;

(b) β is monotone increasing on $E(x_0, x_2)$;

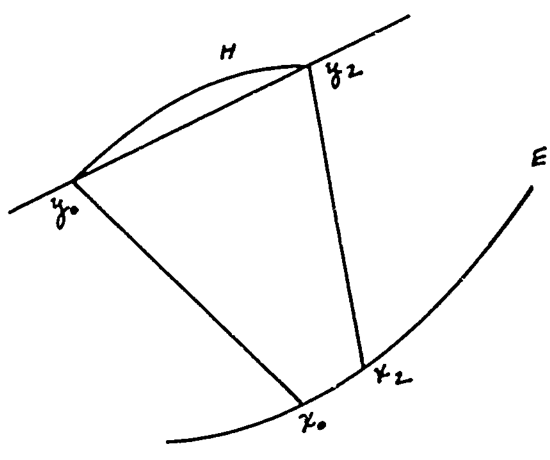
(c) $H(x_0, x_2)$ is E-concave.



H is E-concave



H is neither E-concave
nor E-convex



H is E-convex

Figure 6.3

Proof Let x, x_1 be in $E(x_0, x_2)$; if (a) holds, then with the notation of Fig. 6.1, $\gamma_1 \geq \gamma$. That is, $\beta(x) = \gamma + \text{const} \leq \gamma_1 + \text{const} = \beta(x_1)$. Thus (a) implies (b). To establish the next implication let, for $0 < \rho \leq g(x_0)$, B_ρ be the union of B and the segments $[f(x), z] \subset [f(x), x]$ for $x \in E(x_0, x_2)$ and $d(z, f(x)) = \rho$. The boundary of B_ρ is then $[y_0, y_2] \cup [y_2, z_2] \cup H_\rho \cup [z_0, y_0]$ where H_ρ is an arc of C_ρ. The rays $p(x)$ are normals to H_ρ. If we assume (b), B_ρ has then a line of support at each point of its boundary and is hence convex. $B = \cap B_\rho$ is consequently also convex. Moreover $d(x, [y_0, y_2]) \geq d(x, B_\rho) = g(x) - \rho$ for all $\rho > 0$. At the limit $d(x, [y_0, y_2]) \geq d(x, B)$. Thus (c) holds whenever (b) does.

If we now assume (c) we know that π_B shrinks on the whole plane [11] because B is convex. Further, since $d(x, [y_0, y_2]) \geq d(x, B)$ for $x \in E(x_0, x_2)$, $f(x) \in \pi_B x$. Thus f shrinks on $E(x_0, x_2)$ and (a) is proven.

Theorem 6.5 With the notation and under the assumption of Theorem 6.4, the following three statements are equivalent:

(a) f expands on $E(x_0, x_2)$;

(b) β is monotone decreasing on $E(x_0, x_2)$;

(c) $H(x_0, x_2)$ is E-convex.

Proof The equivalence of (a) and (b) follows from Theor. 6.4 and the continuity of β. The equivalence of (b) and (c) may be similarly established, after observing that, in the proof of Theor. 6.4, it is the monotonicity of β which assures, and is indeed equivalent to, the

convexity of B.

An immediate application of Lemma 6.3 or of Theorems 6.4 and 6.5 (together with the known result that the boundary of a convex, bounded set is rectifiable) yield:

Theorem 6.6 Let $E(x_0, x_2)$ be rectifiable and be the union of finitely many arcs, on every one of which β is monotone. Then $H(x_0, x_2)$ is also rectifiable.

It can be shown by examples that the assumption on β is not necessary. It is even conjectured that H be rectifiable whenever E is.

7. Assuming Differentiability

In this section we will use the notation introduced above and assume $H = H(x_0, x_2)$ to have a continuous tangent $t(y)$ at every point y. Remember that then $tf(x)$ and $p(x)$ are orthogonal; we assume $tf(x)$ so oriented that the angle from it to $p(x)$ is $+\pi/2$.

We will denote by $\lim_{x_1 \to x^+} g(x)$, for any function g defined on $E(x_0, x_2)$ the limit, if it exists, of g when x_1 tends to x from the right, that is with x_1 following x on E.

Theorem 7.1 If H has curvature $c(y)$ at $y = f(x)$, then $\lim_{x_1 \to x^+} \frac{d(x, x_1)}{d(y, y_1)}$, where $y_1 = f(x_1)$, exists and is given by $\frac{1 + c(y)\, g(x)}{\sin \alpha(x)}$. Further that limit is zero iff x is the center of curvature at y.

Proof Assume $c(y) \geq 0$, that is the situation of Fig. 7.1 then

$$d(z, x_1) \sin\varphi = d(x, x_1) \sin\gamma$$

$$\frac{d(z, x_1)}{d(y, y_1)} = \frac{d(y_1, w) + g(x_1)}{d(y_1, w)} = 1 + \frac{g(x_1)}{d(y_1, w)}.$$

Combining the two relations we obtain

$$\frac{d(x, x_1)}{d(y, y_1)} = \frac{\sin\varphi}{\sin\gamma}\left(1 + \frac{g(x_1)}{d(y_1, w)}\right).$$

Observe now that when $x_1 \to x$, then also

$$\varphi \to \pi/2, \quad \gamma \to \alpha(x)$$

$$g(x_1) \to g(x), \quad \frac{1}{d(y_1, w)} \to |c(y)|.$$

The last convergence follows from the fact that w tends to the center of curvature at y. Our result follows. If $c(y) \leq 0$, the proof is similar.

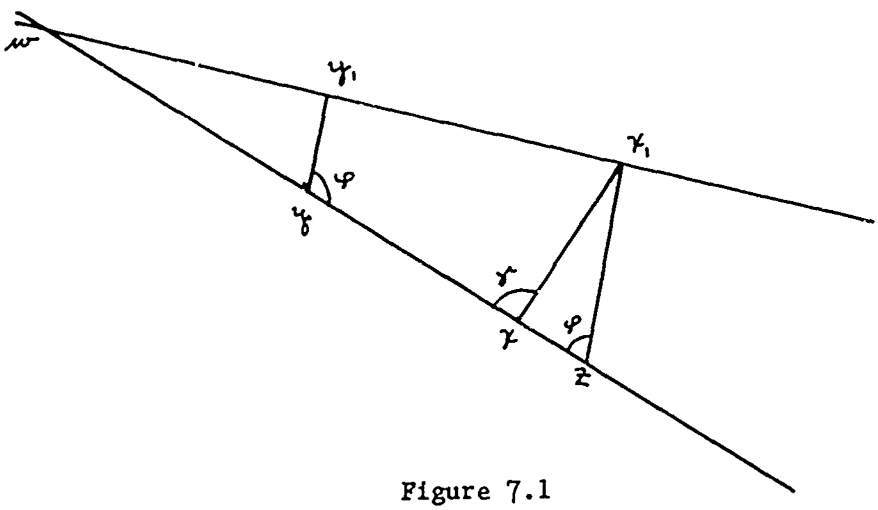


Figure 7.1

To complete the proof observe that $1 + c(y)g(x) = 0$ is equivalent to $c(y) = -\frac{1}{g(x)}$; that is x is the center of curvature.

If $E(x_0, x_2)$ and $H(x_0, x_2)$ are rectifiable, Theor. 7.1 yields also the value of

$$\lim_{x_1 \to x^+} \frac{\text{arc length } (x, x_1)}{\text{arc length } (y, y_1)}$$

since, on any rectifiable curve,

$$\lim \frac{\text{arc length } (x, x_1)}{d(x, x_1)} = 1.$$

We shall denote by a dot the <u>derivative along</u> E (if it exists) of any function defined there. For instance,

$$\dot{\beta}(x) = \lim_{x_1 \to x^+} \frac{\beta(x_1) - \beta(x)}{d(x_1, x)}.$$

Then if $\tau(x)$ denotes the angle from ℓ to $t(x)$, $\dot{\tau}(x)$ is the curvature $c(x)$ of E at x (if it exists).

<u>Theorem 7.2</u> Assume $E(x_0, x_2)$ and $H = H(x_0, x_2)$ rectifiable, and assume that H has curvature at $y = f(x)$. Then, if $\dot{\beta}(x)$ exists,

$$c(y) = \frac{\dot{\beta}(x)}{\sin \alpha(x) - g(x)\dot{\beta}(x)}.$$

<u>Proof</u> We have

$$\frac{\beta(x_1) - \beta(x)}{d(x_1, x)} = \frac{\beta(x_1) - \beta(x)}{d(y_1, y)} \; \frac{d(y_1, y)}{d(x_1, x)}.$$

Since $\beta(x)$ is the angle from ℓ to the normal $p(x)$ to H at $f(x)$, the first factor in the right hand side tends, under our assumptions, to

$c(y)$ (when $x_1 \to x^+$). Thus, at the limit, using Theor. 7.1,

$$\dot\beta(x) = c(y)\,\frac{\sin\alpha(x)}{1 + c(y)\,q(x)}$$

which is equivalent to the desired result.

In view of the last theorem, it is important to establish criteria for the existence of $\dot\beta(x)$.

Lemma 7.3 Under the assumptions of Theor. 7.2, $\dot\beta(x)$ exists when x is not the center of curvature to H at y.

Proof From the proof of Theor. 7.2 and from Theor. 7.1.

Lemma 7.4 The existence of any two of $\dot\alpha$, $c = \dot\tau$, $\dot\beta$ implies the existence of the third. If they exist, then $\dot\beta = \dot\alpha + c$.

Proof $\beta = \alpha + \tau$.

Lemma 7.5 Under the assumptions of Theor. 7.2, and if x is not the center of curvature to H at y, then $\dot\alpha(x)$ exists iff $c(x)$ exists.

Proof Assume $c(x)$ to exist. Since $\beta(x) = \alpha(x) + \tau(x)$,

$$\frac{\alpha(x_1) - \alpha(x)}{d(x_1, x)} = \frac{\beta(x_1) - \beta(x)}{d(y_1, y)}\;\frac{d(y_1, y)}{d(x_1, x)} - \frac{\tau(x_1) - \tau(x)}{d(x_1, x)}\,.$$

Under our assumptions, when $x_1 \to x^+$ the right hand side converges to $c(y)\,\frac{\sin\alpha(x)}{1 + c(y)\,q(x)} - c(x)$. Thus the left hand side converges also, yielding $\dot\alpha(x)$. The converse is established in a similar fashion.

We now assume $E(x_0, x_2)$ to be rectifiable and denote by

$x(s) = (\xi_1(s), \xi_2(s))$ the parametric representation of $E(x_0, x_2)$ in some cartesian coordinate system having ℓ for first axis, where s is the arc length along $E(x_0, x_2)$. Let us set

$$\beta_0(s) = \beta(x(s)).$$

If we denote by $\dot{\beta}_0$ the derivative of β_0 with respect to s , we will have

$$\dot{\beta}_0(s) = \dot{\beta}(x(s)).$$

There will be thus no confusions possible in using a dot to denote differentiation with respect to s . We shall also set

$$g_0(s) = g(x(s)) \qquad \tau_0(s) = \tau(x(s))$$
$$\alpha_0(s) = \alpha(x(s))$$

and remember that $g^{+}(x)$ exists and is given by $-\cos\alpha(x)$ (see [3]); in our situation $\dot{g}_0(s) = -\cos\alpha_0(s)$.

The point $y(s) = f(x(s))$ may be represented as follows:

$$y(s) = (\eta_1(s), \eta_2(s)) = (\xi_1(s) + g_0(s)\cos\beta_0(s), \ \xi_2(s) + g_0(s)\sin\beta_0(s)).$$

Assuming $\dot{\beta}_0$ to exist, we see that $\dot{y}$ exists as well and

$$\dot{y}(s) = (\dot{\xi}_1(s) + \dot{g}_0(s)\cos\beta_0(s) - g_0(s)\dot{\beta}_0(s)\sin\beta_0(s),$$
$$\dot{\xi}_2(s) + \dot{g}_0(s)\sin\beta_0(s) + g_0(s)\dot{\beta}_0(s)\cos\beta_0(s)).$$

If H is rectifiable then its length may be computed as

$$\int_0^{\lambda} \sqrt{\dot{\eta}_1^2(s) + \dot{\eta}_2^2(s)}\, ds \ , \qquad \lambda = \text{length of } E(x_0, x_2).$$

We have

$$\dot{\eta}_1^2(s) + \dot{\eta}_2^2(s) = 1 + \dot{g}_0^2(s) + g_0^2(s)\dot{\beta}_0^2(s)$$

$$+ 2\cos\tau_0(s)\left[\dot{g}_0(s)\cos\beta_0(s) - g_0(s)\dot{\beta}_0(s)\sin\beta_0(s)\right]$$

$$+ 2\sin\tau_0(s)\left[\dot{g}_0(s)\sin\beta_0(s) + g_0(s)\dot{\beta}_0(s)\cos\beta_0(s)\right]$$

$$= 1 + \cos^2\alpha_0(s) + g_0^2(s)\dot{\beta}_0^2(s)$$

$$+ 2\dot{g}_0\cos\alpha_0(s) - 2g_0(s)\dot{\beta}_0(s)\sin\alpha_0(s)$$

$$= \left[g_0(s)\dot{\beta}_0(s) - \sin\alpha_0(s)\right]^2 .$$

We have thus proven:

<u>Theorem 7.6</u> Let $E(\chi_0,\chi_2)$ and $H = H(\chi_0,\chi_2)$ be rectifiable and assume $\dot{\beta}$ to exist on $E(\chi_0,\chi_2)$. Then, with the notation just introduced, the length of H is given by

$$\int_0^{\lambda} \left| g_0(s)\dot{\beta}_0(s) - \sin\alpha_0(s)\right| ds.$$

8. On the Convexity of F

We give here three necessary and sufficient conditions for F to be convex. We shall say that π <u>expands</u> iff from $y \in \pi x$, $y' \in \pi x'$ follows $d(x,x') \leq d(y,y')$; as above, for $y \in \pi x$, we shall call

projecting ray the ray $[x, y \rightarrow)$; and on every edge E we consider the mapping β as well as a mapping β' which associates to $x \in E$ the angles $\beta(x)$, $\beta'(x)$ from a fixed line ℓ to the projecting rays $[x, y \rightarrow)$, $[x, y' \rightarrow)$ respectively, where y is on the left, y' on the right of $t(x)$. We then have:

Theorem 8.1 The following four statements are equivalent:

(a) F is convex;

(b) no two projecting rays intersect;

(c) π expands;

(d) on every edge E , β is monotone decreasing and β' monotone increasing.

Proof Let $[x, y \rightarrow)$, $[x', y' \rightarrow)$ be two projecting rays intersecting at w and let $[y', z]$, $[y, z']$ be tangents to the circles of support of center x and x' respectively (Fig. 8.1). Since $z, z' \in F$, if (a) holds the triangles $x y' z$ and $x' y z'$ are contained in F . But necessarily either y is interior to $x y' z$ or y' is interior to $x' y z'$, an impossibility since $y, y' \in C$. Thus (a) implies (b); the implication (b)$\rightarrow$(c) is obvious. Theorems 6.4 and 6.5 yield the next implication (c)$\rightarrow$(d). Finally, to close the loop and show (d)$\rightarrow$(a) one can use reasonings parallel to those used to prove the implication (b)$\rightarrow$(c) of Theor. 6.4.

It may be interesting to compare the equivalence " F is convex iff π expands" with Phelps's result: " A is convex iff π shrinks" [11].

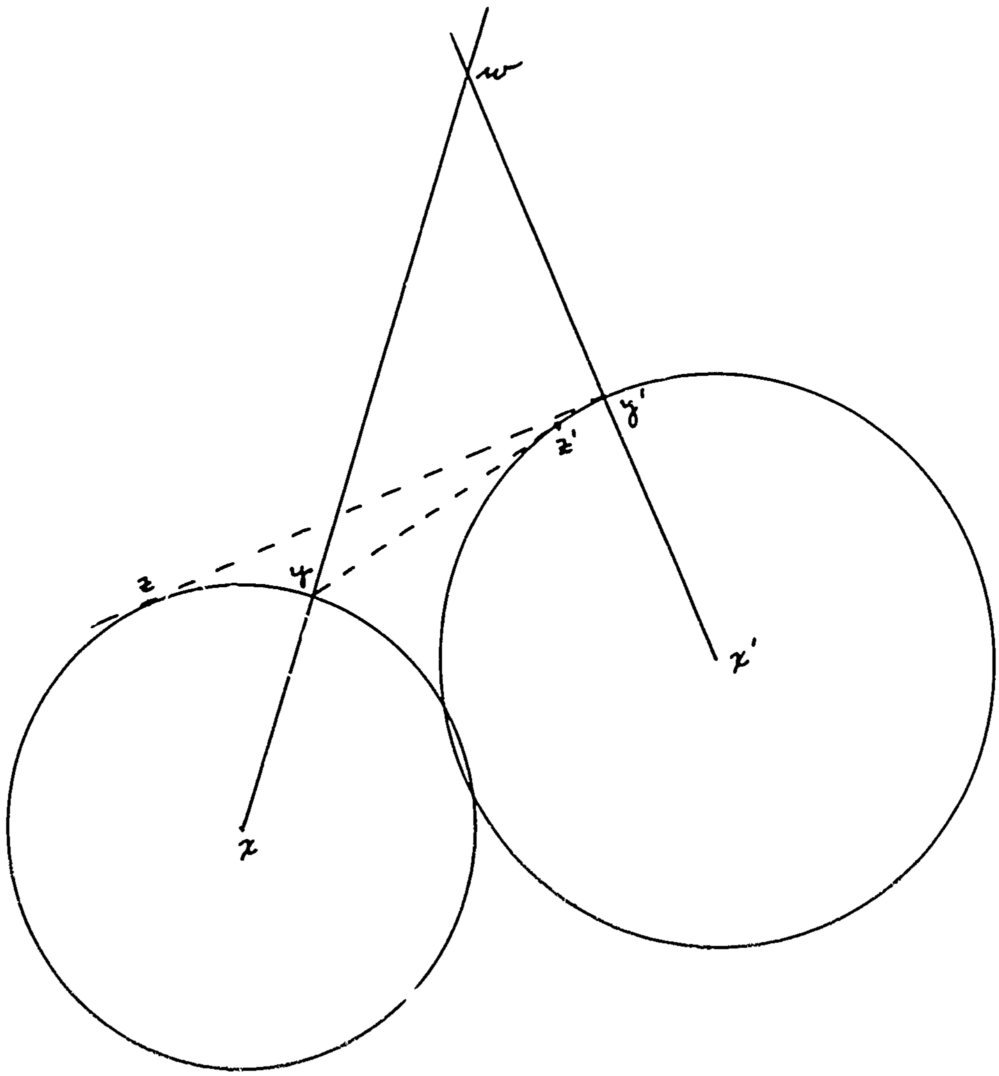


Figure 8.1

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13. ABSTRACT

Part One of this report is a brief summary of the research performed under the contract. Part Two presents in 8 sections a study of the skeletal graph G of a silhouette F and some of the relations between the properties of G and the shape of F.

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